Proof that

~~$A(x, \xi, y)$~~ $A(x,y)$ real $\Rightarrow E_{f_p}(\xi,\eta) = E^*_{f_p}(-\xi,-\eta)$

$$E_{f_p}(\xi,\eta) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} A(x,y)\, e^{-i2\pi(x\xi + y\eta)}\, dx\, dy$$

$$= \int_{y=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty} A(x,y)\cos(2\pi x\xi) - iA(x,y)\sin(2\pi x\xi)\, dx\right) e^{-i2\pi y\eta}\, dy$$

$$\int_{y=-\infty}^{\infty} P_{Re}(\xi,y)\, e^{-i2\pi y\eta}\, dy - i\int_{y=-\infty}^{\infty} P_{im}(\xi,y)\, e^{-i2\pi y\eta}\, dy$$

$$= \int_{y=-\infty}^{\infty} P_{Re}(\xi,y)\cos(2\pi y\eta)\, dy - i\int_{-\infty}^{\infty} P_{Re}(\xi,y)\sin(2\pi y\eta)\, dy$$

$$- i\int_{x=-\infty}^{\infty} P_{im}(\xi,y)\cos(2\pi y\eta)\, dy - i(-i)\int_{-\infty}^{\infty} P_{im}(\xi,y)\sin(2\pi y\eta)\, dy$$

where $P_{Re}(\xi,y) = \int_X A(x,y)\cos(2\pi x\xi)\, dx$ and $P_{Im}(\xi,y) = \int_X A(x,y)\sin(2\pi x\xi)\, dx$

$$E_{f_p}(-\xi, -\eta) = \int_y P_{Re}(-\xi, y)\cos(2\pi y\eta)\,dy + \int_y P_{Im}(-\xi, y)\sin(2\pi y\eta)\,dy$$

$$+ i\int_y P_{Re}(-\xi, y)\sin(2\pi y\eta)\,dy - i\int_y P_{Im}(-\xi, y)\cos(2\pi y\eta)\,dy$$

using

$$P_{Re}(-\xi, y) = P_{Re}(\xi, y) \qquad P_{Im}(-\xi, y) = -P_{Im}(\xi, y)$$

$$\Rightarrow E_{f_p}(-\xi, -\eta) = \int_y P_{Re}(\xi, y)\cos(2\pi\xi\eta)\,dy - \int_y P_{Im}(\xi, y)\sin(2\pi y\eta)\,dy$$

$$+ i\int_y P_{Re}(\xi, y)\sin(2\pi y\eta)\,dy + i\int_y P_{Im}(\xi, y)\cos(2\pi y\eta)\,dy$$

$$= Re\{E(\xi, \eta)\} - i\, Im\{E(\xi, \eta)\}$$

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if $A(x, y) = A(-x, y)$, then $P_{Re}(\xi, y) = 2\int_{x=0}^{\infty} A(x, y)\cos(2\pi x\xi)\,dx$

and $P_{Im}(\xi, y) = 0$

$$\Rightarrow E(\xi, -\eta) = \int_y P_{Re}(\xi, y)\cos(2\pi y\eta)\,dy + i\int_y P_{Re}(\xi, y)\sin(2\pi y\eta)\,dy$$

$$= Re\{E(\xi, \eta)\} - i\, Im\{E(\xi, \eta)\}$$

Therefore, for real $A(x,y)$ and $A(x,y) = A(-x,y)$

$$Im\{E_{fp}(\xi,\eta)\} = -Im\{E_{fp}(-\xi,-\eta)\} \text{ and } Im\{E_{fp}(\xi,\eta)\} = -Im\{E_{fp}(\xi,-\eta)\}$$
$$\text{and } Im\{E_{fp}(-\xi,\eta)\} = Im\{E_{fp}(\xi,\eta)\}$$

$$Re\{E_{fp}(\xi,\eta)\} = Re\{E_{fp}(-\xi,-\eta)\} = Re\{E_{fp}(\xi,-\eta)\}$$

The DFT need only be evaluated for one quadrant of the image plane to determine the E field over the full image.

Inverse DFT to propagate to the next (Lyot) pupil plane, after applying focal plane mask $M_{fp}(\xi,\eta)$, so that

$$E_{fpm}(\xi,\eta) = M_{fp}(\xi,\eta)\, E_{fp}(\xi,\eta)$$

and $M_{fp}(\xi,\eta)$ is real and two-fold symmetric.

$$E_{Lp}(u,v) = \int_{\xi=-\infty}^{\infty}\int_{\eta=-\infty}^{\infty} E_{fpm}(\xi,\eta)\, e^{i2\pi(u\xi+v\eta)}\, d\xi\, d\eta$$

$$= \int_{\eta=-\infty}^{\infty}\left[\int_{\xi=-\infty}^{\infty} E_{fpm}(\xi,\eta)\, e^{i2\pi\xi u}\, d\xi\right] e^{i2\pi v\eta}\, d\eta$$

$$= \int_{\eta=-\infty}^{\infty} P_{Lp}(u,\eta)\, e^{i2\pi v\eta}\, d\eta$$

$$\text{where } P_{Lp}(u,\eta) = \int_{\xi=-\infty}^{\infty} E_{fpm}(\xi,\eta)\, e^{i2\pi\xi u}\, d\xi$$

But, both the real and imaginary parts of $E_{fpm}(\xi, \eta)$ are even symmetric in ξ. So

$$P_{Lp}(u, \eta) = 2 \int_{\xi=0}^{\infty} E_{fpm}(\xi, \eta) \cos(2\pi \xi u)\, d\xi$$

and $$E_{Lp}(u,v) = 2 \int_{\eta=-\infty}^{\infty} \left[\int_{\xi=0}^{\infty} E_{fpm}(\xi, \eta) \cos(2\pi \xi u)\, d\xi \right] e^{i2\pi v\eta}\, d\eta$$

$$= 2 \int_{\eta=-\infty}^{\infty} \int_{\xi=0}^{\infty} E_{fpm}(\xi, \eta) \cos(2\pi \xi u)\, d\xi \, \cos(2\pi \eta v)\, d\eta$$

$$+ 2i \int_{\eta=-\infty}^{\infty} \int_{\xi=0}^{\infty} E_{fpm}(\xi, \eta) \cos(2\pi \xi u)\, d\xi \, \sin(2\pi \eta v)\, d\eta$$

The real part of the inner integral is even symmetric in η and the imaginary part is odd symmetric in η.

or equivalently $\mathrm{Re}\{P_{Lp}(u, \eta)\} = \mathrm{Re}\{P_{Lp}(u, -\eta)\}$ and

$$\mathrm{Im}\{P_{Lp}(u, \eta)\} = -\mathrm{Im}\{P_{Lp}(u, -\eta)\}$$

Using the real and imaginary parts of $P_{Lp}(u,\eta)$ separately,

$$E_{Lp}(u,v) = 2\int_{\eta=-\infty}^{\infty} \mathrm{Re}\{P_{Lp}(u,\eta)\}\cos(2\pi\eta v)\,d\eta$$

$$+(i)2i\int_{\eta=-\infty}^{\infty} \mathrm{Im}\{P_{Lp}(u,\eta)\}\sin(2\pi\eta v)\,d\eta$$

$$= \underbrace{4\int_{\eta=0}^{\infty} \mathrm{Re}\{P_{Lp}(u,\eta)\}\cos(2\pi\eta v)\,d\eta}_{\text{even function of } u \text{ and } v} - \underbrace{4\int_{\eta=0}^{\infty} \mathrm{Im}\{P_{Lp}(u,\eta)\}\sin(2\pi\eta v)\,d\eta}_{\text{odd function of } V,\ \text{even func of } u}$$

So the Lyot plane E field is real, and symmetric across the vertical axis, like the telescope pupil E field.