Lyot coronagraph Fourier propagation (1)

Goodman eqn 5-15 is the Fraunhofer diffraction of an "apodized" ~~aperture~~ lens :

$$U_f(u,v) = \frac{\exp\left[\frac{ik}{2f}(u^2+v^2)\right]}{i\lambda f} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_l(x,y) \exp\left[\frac{-i2\pi}{\lambda f}(xu+yv)\right] dx\, dy$$

let the apodization be $A(x,y)$ instead of $U(x,y)$

integration limits are $\left[\frac{-D}{2}, \frac{D}{2}\right]$

Leaving out phase factor for now

$$U_f(u,v) = \frac{1}{i\lambda f} \int_{-\frac{D}{2}}^{\frac{D}{2}}\int_{-\frac{D}{2}}^{\frac{D}{2}} A(x,y) \exp\left[\frac{-i2\pi}{\lambda f}(xu+yv)\right] dx\, dy$$

$$x' = \frac{x}{D} \quad y' = \frac{y}{D} \qquad \xi = \frac{uD}{\lambda f} \quad \eta = \frac{vD}{\lambda f}$$

$$dx' = \frac{dx}{D} \quad dy' = \frac{dy}{D}$$

$$U_f(\xi,\eta) = \frac{D^2}{i\lambda f} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} A'(x',y') \exp\left[-i2\pi(x'\xi + y'\eta)\right] dx'\, dy'$$

$$\left(\text{if } A(x,y) = \Pi\left(\frac{x}{W_x}\right)\Pi\left(\frac{y}{W_y}\right) \text{ then } A'(x',y') = \Pi\left(\frac{x'D}{W_x}\right)\Pi\left(\frac{y'D}{W_y}\right)\right)$$

$$= \frac{D^2}{i\lambda f} \mathcal{F}\{A'(x',y')\}$$

For rectangular aperture $\Pi\left(\frac{x'D}{w_x}\right)\Pi\left(\frac{y'D}{w_y}\right)$,

$$U_f(\xi,\eta) = \frac{\cancel{D^2}}{i\lambda f}\frac{w_x w_y}{\cancel{D^2}}\,\mathrm{sinc}\left(\xi\frac{w_x}{D}\right)\mathrm{sinc}\left(\eta\frac{w_y}{D}\right)$$

For circular aperture $\Pi_{circ}(r') = \Pi_{circ}\left(\frac{r}{D}\right)$,

agrees with Goodman eqn 4-28

$$U_f(\rho) = \frac{1}{i\lambda f}\mathcal{B}\left\{\Pi_{circ}\left(\frac{r}{2R}\right)\right\}$$

$$= \frac{2\pi}{i\lambda f}\int_0^R r J_0(2\pi r\rho)\,dr$$

let $\Theta = 2\pi r\rho$, $r = \frac{\Theta}{2\pi\rho}$ $dr = \frac{d\Theta}{2\pi\rho}$

$$\frac{2\pi}{i\lambda f}\int_0 \frac{\Theta}{2\pi\rho}J_0(\Theta)\frac{d\Theta}{2\pi\rho}$$

$r=R \Rightarrow \Theta = 2\pi R\rho$

$$\frac{1}{i\lambda f}\frac{1}{2\pi\rho^2}\int_0^{2\pi R\rho}\Theta J_0(\Theta)\,d\Theta = \frac{1}{i\lambda f}\frac{2\pi R\rho}{2\pi\rho^2}J_1(2\pi R\rho)$$

$$\frac{1}{i\lambda f}\frac{R}{\rho}J_1(2\pi R\rho)$$

$$\Theta' = \frac{2\pi r\rho}{2R} = \frac{\Theta}{2R} \Rightarrow d\Theta' = \frac{d\Theta}{2R}$$

limit $\Theta = 2\pi R\rho \Rightarrow \Theta' = \pi\rho$

$$\frac{i}{\lambda f}\frac{4R^2}{2\pi\rho^2}\int_0^{\pi\rho}\Theta' J_0(2R\Theta')\,d\Theta'$$

$$\frac{i}{\lambda f}\frac{1}{2\pi\rho^2}\int_0^{\pi\rho}2R\Theta' J_0(2R\Theta')\,2R\,d\Theta'$$

$$\frac{i}{\lambda f}\frac{\pi\rho}{2\pi\rho^2}$$

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$$U_f(\rho) = \frac{2\pi}{i\lambda f} \int_0^{D/2} r J_0\left(\frac{2\pi r \rho}{\lambda f}\right) dr$$

$$r' = \frac{r}{D} \quad dr' = \frac{dr}{D} \quad \rho' = \frac{\rho D}{\lambda f}$$

$$U_f(\rho') = \frac{2\pi}{i\lambda f} \int_0^{1/2} D^2 r' J_0(2\pi r' \rho')\, dr'$$

$$\frac{2\pi D^2}{i\lambda f} \int_0^{1/2} r' J_0(2\pi r' \rho')\, dr'$$

$$\theta = 2\pi r' \rho' \quad r' = \tfrac{1}{2} \Rightarrow \theta = \pi\rho'$$

$$\frac{2\pi D^2}{(2\pi\rho')^2\, i\lambda f} \int_0^{\pi\rho'} \theta J_0(\theta)\, d\theta$$

$$\frac{\cancel{\pi\rho'}\, \cancel{2\pi} D^2}{i\lambda f\, \cancel{(2\pi\rho')}(2\pi\rho')} J_1(\pi\rho')$$

$$\frac{2\pi D}{\lambda f\, 2}\rho = \pi\rho'$$

$$U_f(\rho') = \frac{D^2}{i\lambda f\, 2\rho'} J_1(\pi\rho')$$

$$\rho' \to \rho$$

$$U_f(\rho) = \frac{D^2 J_1(\pi\rho)}{i 2\lambda f \rho} = \frac{2\pi\left(\frac{D}{2}\right)^2 J_1(\pi\rho)}{i\lambda f \;\pi}$$

$$= \frac{A}{i\lambda f} \cdot \frac{2J_1(\pi\rho)}{\pi\rho}$$

(agrees with Goodman eqn 4-31)

square aperture, diameter D

(4)

$$U_f(\xi,\eta) = \frac{D^2}{i\lambda f}\,\mathrm{sinc}(\xi)\,\mathrm{sinc}(\eta)$$

focal plane stop $\alpha_0\left(\frac{\lambda f}{D}\right)$ physical diameter, $\alpha = \alpha_0 \frac{\lambda_0}{\lambda}$

$$1 - M(\xi,\eta) = 1 - \Pi\left(\frac{\xi}{\alpha}\right)\Pi\left(\frac{\eta}{\alpha}\right)$$

$$= 1 - \Pi\left(\frac{uD}{f\lambda\alpha}\right)\Pi\left(\frac{vD}{f\lambda\alpha}\right)$$

$$= 1 - \Pi\left(\frac{uD}{f\lambda_0\alpha_0}\right)\Pi\left(\frac{vD}{f\lambda_0\alpha_0}\right)$$

$$\hat{M}(x,y) = \frac{f^2\lambda_0^2\alpha_0^2}{D^2}\,\mathrm{sinc}\left(\frac{f\lambda_0\alpha_0}{D}x\right)\mathrm{sinc}\left(\frac{f\lambda_0\alpha}{D}y\right)$$

without changing variables ~~and at one wavelength~~,

$$\hat{M}(x,y) = \alpha^2\,\mathrm{sinc}(\alpha x)\,\mathrm{sinc}(\alpha y)$$

$$= \frac{\sin(\alpha\pi x)}{\pi x}\,\frac{\sin(\alpha\pi y)}{\pi y}$$

$$\Psi_C^-(x,y) = \cancel{\Psi_A(x,y) - \Psi_A(x,y)*}$$

$$\Psi_A(x,y) * \left[\delta(x,y) - \hat{M}(x,y)\right]$$

$$= \Psi_A(x,y) - \Psi_A(x,y) * \frac{\sin(\alpha\pi x)}{\pi x}\,\frac{\sin(\alpha\pi y)}{\pi y}$$

$$U_f(\rho) = \frac{2\pi}{i\lambda f}\int_0^R r J_0(2\pi r\rho)\,dr$$

$$r' = \frac{r}{2R} \qquad dr' = \frac{dr}{2R} \qquad \rho' = \rho\frac{2R}{\lambda f}$$

$$\frac{2\pi}{i\lambda f}\int_0^{1/2} 2R\,r' J_0(2\pi\, 2R r'\rho)\, 2R\,dr'$$

$$\frac{2\pi\, 4R^2}{i\lambda f}\int_0^{1/2} r' J_0(4\pi R r'\rho)\,dr'$$

$$\theta = 4\pi R r'\rho \qquad r' = \tfrac{1}{2} \Rightarrow \theta = 2\pi R\rho$$

$$\frac{2\pi(2R)^2}{i\lambda f}\int_0^{2\pi R\rho} \frac{\theta}{4\pi R\rho}\frac{1}{4\pi R\rho} J_0(\theta)\,d\theta$$

$$\frac{2\pi(2R)^2}{i\lambda f(2\pi)^2(2R)^2\rho^2}\int_0^{2\pi R\rho} \theta J_0(\theta)\,d\theta$$

$$\frac{\cancel{2\pi}\cancel{(2R)^2}\,\cancel{2\pi}\,R\cancel{\rho}}{i\lambda f\cancel{(2\pi)^2}\cancel{(2R)^2}\rho^{2}} J_1(2\pi R\rho)$$

$$\frac{R}{i\lambda f\rho} J_1(2\pi R\rho)$$

Circular case

$$1 - M(\rho) = 1 - \Pi\left(\frac{\rho}{\alpha}\right)$$

and $$\hat{M}(r) = \pi\left(\frac{\alpha}{2}\right)^2 \frac{2 J_1(\alpha \pi r)}{\alpha \pi r}$$

$$= \frac{\alpha J_1(\alpha \pi r)}{2r}$$

$$\Psi_c^-(r) = \Psi_A(r) - \Psi_A(r) * \frac{\alpha J_1(\alpha \pi r)}{2r}$$

symmetry for square ap

$$U_{f1}(\xi,\eta) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} A_x(x) A_y(y)\, e^{-i2\pi x\xi}\, e^{-i2\pi y\eta}\, dx\, dy$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} A_x(x) e^{-i2\pi x\xi}\, dx \int_{-\frac{1}{2}}^{\frac{1}{2}} A_y(y) e^{-i2\pi y\eta}\, dy$$

if A_x and A_y are even

$$U_{f1}(\xi,\eta) = \underbrace{\left(2\int_0^{\frac{1}{2}} A_x(x)\cos(2\pi x\xi)\, dx\right)}_{= U_{f1,x}(\xi)} \underbrace{\left(2\int_0^{\frac{1}{2}} A_y(y)\cos(2\pi y\eta)\, dy\right)}_{= U_{f1,y}(\eta)}$$

$$U_L(u,v) = \mathcal{F}\{U_{f1}(\xi,\eta)\}$$

$$= \mathcal{F}\left\{ U_{f1,x}(\xi)\, U_{f1,y}(\eta) \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{f1,x}(\xi)\, U_{f1,y}(\eta)\, e^{-i2\pi\xi u}\, e^{-i2\pi\eta v}\, d\xi\, d\eta$$